Linear Programming Problem

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Outlines of The Talk

- What is Optimization ?
- Origin of Linear Programming
- Basic Terms
- Formulation and Solution of LPP (Graphical & Simplex)
- Artificial Variable Technique(Big M & Two Phase)
- Types of Solution
- Integer Linear Programming
- Duality In LPP
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- Assignment

What is Optimization ?

Optimization is an important tool in making decisions and in analyzing physical systems. In mathematical terms, an optimization problem is the problem of finding the best solution from among the set of all feasible solutions.

Optimization problems are everywhere in science and engineering and even in our daily life, thinking about how we optimize our way to go to work, the choice of line we stand in at the supermarket or deciding for the education for our children.

(November 8, 1914 – May 13, 2005)
Dr. George B. Dantzig,
"The Father of Linear Programming"

Origin of Linear Programming

Any organization (big or small) has at its disposal men, machines, money and materials, the supply of which may be limited. Supply of resources being limited , the management must find the best allocation of its resources in order to maximize the profit or minimize the loss or utilize the production capacity to the maximum extent . However, this involves a number of problems which can be over by quantitative methods, particularly the linear programming.

The linear programming technique is applicable in problems characterized by the presence of a number of decision variables, each of which can assumes values within a certain range and affect other decision variables.

These variables represent some physical or economic quantities which are of interest to the decision maker and whose domain are governed by some practical limitations of constants. These may be due to availability of resources like men, material or money or may be due to a quality constraint or sales constraint. The problem has also a well defined objective such as minimum cost, maximum profit, highest turn over, maximum capacity utilization and many more.

Linear programming technique indicates the right combination of various decision variables which can be best employed to achieve the objective taking full account of practical limitations with in which the problem must be solved.

The most important feature of the LP technique is the presence of a linearity in the problem. Linear programming was developed as a discipline in the 1940's, motivated initially by the need to solve complex planning problems in wartime operations. Its development accelerated rapidly in the post war period as many industries found valuable uses for linear programming. The founders of the subject are generally regarded as George B. Dantzig, who derived the simplex method in 1947, and John Von Neumann, who established the theory of duality that same year.

The Nobel prize in economics was awarded in 1975 to the mathematician Leonid Kantorovich (USSR) and the economist Tjalling Koopmans (USA) for their contributions to the theory of optimal allocation of resources, in which linear programming played a key role. Many industries use linear programming as a standard tool to allocate a finite set of resources in an optimal way.

Examples of important application areas include airline crew scheduling, shipping or telecommunication networks, oil refining and blending, stock and bond portfolio selection, transportation, energy, planning and manufacturing.

Basic Terms

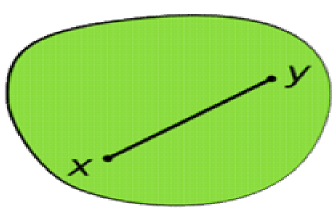
Convex Set:

In Euclidean space, an object is **convex** if for every pair of points within the object, every point on the straight line segment that joins them is also within the object. For example, a solid cube is convex, but anything that is hollow or has a dent in it like a crescent shape, is not convex

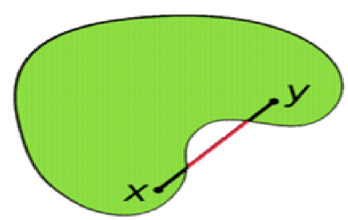
Mathematically,

Let X be any set in the Euclidean space . If x_1 , x_2 ε X, then every point $U = \lambda x_2 + (1-\lambda) x_1$; $0 \leq \lambda \leq 1$ must be in the set X.

Convex set



Non Convex Set



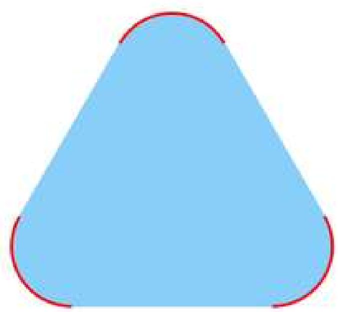
Extreme Point

An extreme point x of a Convex set S in a real vector space is a point in S which does not lie in any open line segment joining two points of S. Intuitively, an extreme point is a "vertex" of *S*.

Mathematically,

There do not exist two distinct points x_1, x_2 ε S such that $x = \lambda x_2 + (1-\lambda) x_1$; $0 < \lambda < 1$.

e.g. The extreme points of a triangle are its vertices.



Programming Problems

The process of determining a particular program is known as programming. The programming problems are the problems which deal with the situations where a number of resources are to be combined to yield one or more products subject to certain restrictions.

The linear programming problems are those programming problems in which relations among the decision variables are linear in both constraints as well in the function to be optimized.

Example: objective function

Find the maximal and minimal value of $z = 3x + 4y$ subject to the following constraints:

$$\begin{cases} x + 2y \le 14 \\ 3x - y \ge 0 \\ x - y \le 2 \end{cases}$$

subject to constraints

and $x, y \ge 0$ non-negetivity condition

Feasible Region

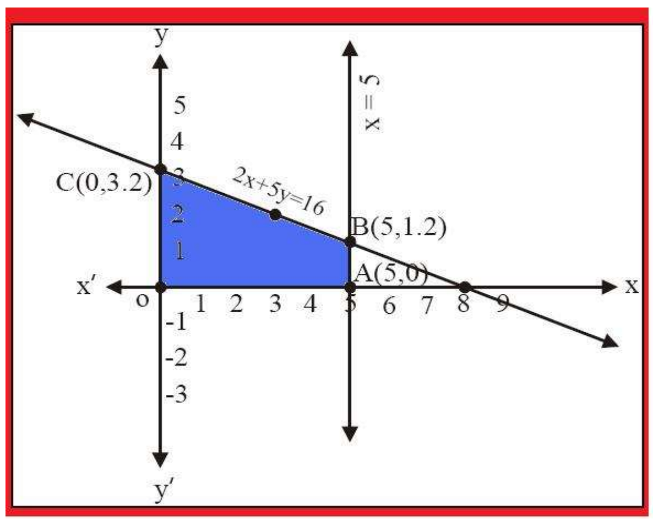
The region in which all the given constraints are satisfied.

For example, feasible region for the following LPP is shown as

Maximise $Z = 2x + 10y$

Subject to the constraints

$2x + 5y \leq 16$, $x \leq 5$ and $x, y \geq 0$.



Formulation of Optimization Problems as LPP

The following points are to be kept in mind to formulate the problem as LPP:

i) Identify and define the decision variables of the problem and express them in terms of algebraic symbols.

ii) Define the objective function as a linear function in terms of decision variables.

iii) State the constraints in terms of decision variables which the objective function should be optimized (i.e. Maximization or Minimization)

iv) Add the non-negative constraints from the consideration that the negative values of the decision variables do not have any valid physical interpretation

Example

A manufacturer produces two types of models M_1 and M_2.Each model of the type M_1 requires 4 hours of grinding and 2 hours of polishing; where as each model of M_2 requires 2 hours of grinding and 5 hours of polishing. The manufacturer has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works 60 hours a week. Profit on M_1 model is Rs.3.00 and on model M_2 is Rs.4.00.Whatever produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he makes maximum profit in a week? Formulate the problem as a LPP.

Formulation

Let X_1 and X_2 be the number of units of M_1 and M_2 model.

The objective function

Max $Z = 3X_1 + 4X_2$

The grinding constraint is given by

$4X_1 + 2X_2 \leq 80$

The polishing constraint is given by

$2X_1 + 5X_2 \leq 180$

Finally we have,

Max $Z = 3X_1 + 4X_2$

Subject to constraints,

$4X_1 + 2X_2 \leq 80$

$2X_1 + 5X_2 \leq 180$ and $X_1, X_2 \geq 0$

Example

A Retired person wants to invest up to an amount of Rs.30,000 in fixed income securities. His broker recommends investing in two Bonds: Bond A yielding 7% and Bond B yielding 10%. After some consideration, he decides to invest at most of Rs.12,000 in bond B and at least Rs.6,000 in Bond A. He also wants the amount invested in Bond A to be at least equal to the amount invested in Bond B. What should the broker recommend if the investor wants to maximize his return on investment? Formulate the problem as LPP.

Formulation

Let X_1 and X_2 be the amount invested in Bonds A and B.
the objective function is to maximize the yielding.
Max $Z = 0.07X_1 + 0.1X_2$ (Objective function)
$X_1 + X_2 \leq 30{,}000$ (Available amount)
$X_1 \geq 6{,}000$ (Minimum amount for investment in bond A)
$X_2 \leq 12{,}000$ (Maximum amount for investment in bond B)
$X_1 \geq X_2$ and $X_1, X_2 \geq 0$
Finally we have the following LPP:
Max $Z = 0.07X_1 + 0.1X_2$
is subjected to three constraints
$X_1 + X_2 \leq 30{,}000$
$X_1 \geq 6{,}000$
$X_2 \leq 12{,}000$
$X_1 - X_2 \geq 0$ and $X_1, X_2 \geq 0$

Example

A city hospital has the following minimal daily requirements for nurses.

Period	Clock time (24 hours day)	Minimum number of nurses required
1	6 a.m. – 10 a.m.	2
2	10 a.m. – 2 p.m.	7
3	2 p.m. – 6 p.m.	15
4	6 p.m. – 10 p.m.	8
5	10 p.m. – 2 a.m.	20
6	2 a.m. – 6 a.m.	6

Nurses report at the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimal number of nurses to be employed so that there will be a sufficient number of nurses available for each period.
Formulate this as a linear programming problem by setting up appropriate constraints and objective function.

Formulation

i)Identify and define the decision variables of the problem

Let X_1, X_2, X_3, X_4, X_5 and X_6 be the number of nurses joining duty at the beginning of periods 1, 2, 3, 4, 5 and 6 respectively.

ii) Define the objective function

Minimize $Z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$

iii) State the constraints to which the objective function should be optimized.

The above objective function is subjected to following constraints.

$X_1 + X_2 \geq 7$, $X_2 + X_3 \geq 15$, $X_3 + X_4 \geq 8$,

$X_4 + X_5 \geq 20$, $X_5 + X_6 \geq 6$ and $X_6 + X_1 \geq 2$

$X_1, X_2, X_3, X_4, X_5, X_6 \geq 0$